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Determinant identities for theta functions

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ABSTRACT

Two proofs of a theta function identity of R.W. Gosper and R. Schroeppel are given. A cubic analogue is presented, and several interesting special cases are noted.

1. Introduction

Let τ be a complex number with positive imaginary part, and let $q = \exp(i\pi\tau)$ so that $|q| < 1$. The Jacobian theta functions may be defined by

$$\theta_1(z|\tau) = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} e^{i(2n+1)z}, \qquad \theta_2(z|\tau) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{i(2n+1)z}, \qquad \theta_3(z|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz},$$

and

$$\theta_4(z|\tau) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2inz}.$$

The following result was stated without proof by R.W. Gosper and R. Schroeppel [8].

Theorem 1.1 *(Gosper and Schroeppel). Let w_1, w_2, w_3, z_1, z_2 and z_3 be complex variables, and consider the 3×3 matrix whose j,k entry is $\theta_r(w_j - z_k|\tau)\theta_s(w_j + z_k|\tau)$, where $r, s \in \{1,2,3,4\}$. Then*

$$\det\big(\theta_r(w_j - z_k|\tau)\theta_s(w_j + z_k|\tau)\big)_{1\leqslant j,k\leqslant 3} = 0. \tag{1}$$

Gosper and Schroeppel observed that many of the identities in the treatise by Whittaker and Watson [11, Chapter 21] are special cases of the identity (1). For example, [8, (4 vars)], cf. [11, Example 5, p. 451]:

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Corollary 1.2. *Let a_1, a_2, b_1 and b_2 be complex variables. Then, for $s = 1, 2, 3$ or 4, we have*

$$\det\big(\theta_1(a_j - b_k|\tau)\theta_s(a_j + b_k|\tau)\big)_{1\leqslant j,k\leqslant 2} = -\theta_1(a_1 - a_2|\tau)\theta_s(a_1 + a_2|\tau)\theta_1(b_1 - b_2|\tau)\theta_s(b_1 + b_2|\tau).$$

Proof. Setting $w_1 = z_1$ and $w_2 = z_2$ in (1) and using the fact that $\theta_1(z|\tau)$ is an odd function of z and so $\theta_1(0|\tau) = 0$, we get

$$\begin{aligned}&\theta_1(z_1 - z_2|\tau)\theta_s(z_1 + z_2|\tau)\theta_1(z_2 - z_3|\tau)\theta_s(z_2 + z_3|\tau)\theta_1(w_3 - z_1|\tau)\theta_s(w_3 + z_1|\tau)\\ &\quad + \theta_1(z_1 - z_3|\tau)\theta_s(z_1 + z_3|\tau)\theta_1(z_2 - z_1|\tau)\theta_s(z_2 + z_1|\tau)\theta_1(w_3 - z_2|\tau)\theta_s(w_3 + z_2|\tau)\\ &\quad - \theta_1(w_3 - z_3|\tau)\theta_s(w_3 + z_3|\tau)\theta_1(z_2 - z_1|\tau)\theta_s(z_2 + z_1|\tau)\theta_1(z_1 - z_2|\tau)\theta_s(z_1 + z_2|\tau)\\ &= 0.\end{aligned}$$

Now cancel the common factor $\theta_1(z_1 - z_2|\tau)\theta_s(z_1 + z_2|\tau)$ and replace (z_1, z_2, z_3, w_3) with (a_2, a_1, b_1, b_2) to complete the proof. $\square$

The remainder of this work is arranged as follows. We give two proofs of Theorem 1.1 in Sections 2 and 3. In Section 4, we give an analogous result for cubic theta functions. Finally, in Section 5 we show that a special case of a 2×2 determinant of cubic theta functions is equivalent to an identity of M. Hirschhorn et al. [9] and how this identity leads to a fundamental result from the theory of elliptic functions in signature 3.

2. First proof of the Gosper–Schroeppel identity

We begin by observing that by the translational properties of theta functions, we have

$$\theta_r(w_j - z_k|\tau)\theta_s(w_j + z_k|\tau) = \gamma\theta_1(W_j - Z_k|\tau)\theta_1(W_j + Z_k|\tau),$$

where the values of W_j, Z_k and γ, which depend on w_j, z_k, q, r and s, can be worked out from [11, p. 464, Ex. 2]. For example, with $r = 4$ and $s = 3$, we have $W_j = w_j + \frac{\pi}{4} + \frac{\pi\tau}{2}$, $Z_k = z_k + \frac{\pi}{4}$ and $\gamma = -iq^{1/2}e^{2iw_j}$. Therefore

$$\begin{aligned}\det\big(\theta_4(w_j - z_k|\tau)\theta_3(w_j + z_k|\tau)\big)_{1\leqslant j,k\leqslant 3} &= \det\big(-iq^{1/2}e^{2iw_j}\theta_1(W_j - Z_k|\tau)\theta_1(W_j + Z_k|\tau)\big)_{1\leqslant j,k\leqslant 3}\\ &= iq^{3/2}e^{2i(w_1+w_2+w_3)}\det\big(\theta_1(W_j - Z_k|\tau)\theta_1(W_j + Z_k|\tau)\big)_{1\leqslant j,k\leqslant 3}.\end{aligned}$$

Consequently, it suffices to prove Theorem 1.1 in the case $r = s = 1$.

By [11, p. 465], the function θ_1 satisfies the functional equations

$$\theta_1(z + \pi|\tau) = -\theta_1(z|\tau) \quad \text{and} \quad \theta_1(z + \pi\tau|\tau) = -q^{-1}e^{-2iz}\theta_1(z|\tau).$$

It follows that

$$\theta_1(w - z - \pi|\tau)\theta_1(w + z + \pi|\tau) = \theta_1(w - z|\tau)\theta_1(w + z|\tau) \tag{2}$$

and

$$\theta_1(w - z - \pi\tau|\tau)\theta_1(w + z + \pi\tau|\tau) = q^{-2}e^{-4iz}\theta_1(w - z|\tau)\theta_1(w + z|\tau). \tag{3}$$

Fix w_1, w_2, w_3, z_2 and z_3 and consider the determinant as a function of z_1, i.e., let

$$F(z_1) = \det\big(\theta_1(w_j - z_k|\tau)\theta_1(w_j + z_k|\tau)\big)_{1\leqslant j,k\leqslant 3}.$$

Furthermore, let

$$G(z_1) = \theta_1(z_2 + z_1|\tau)\theta_1(z_2 - z_1|\tau).$$

Then (2), (3) and elementary properties of the determinant imply

$$F(z_1 + \pi) = F(z_1), \qquad F(z_1 + \pi\tau) = q^{-2}e^{-4iz_1}F(z_1). \tag{4}$$

Clearly, by (2) and (3) we also have

$$G(z_1 + \pi) = G(z_1), \qquad G(z_1 + \pi\tau) = q^{-2}e^{-4iz_1}G(z_1). \tag{5}$$

If $z_1 = \pm z_2$ or $z_1 = \pm z_3$ then the matrix in the definition of $F(z_1)$ will have two identical columns, and so $F(z_1) = 0$. From this, and using (4), it follows that the function $F(z_1)$ has zeros at $z_1 = \pm z_2 + m\pi + n\pi\tau$ and at $z_1 = \pm z_3 + m\pi + n\pi\tau$, where m and n are any integers (and possibly at other points, too). It is known [11, p. 470] that the zeros of $\theta_1(z|\tau)$ are all simple and occur precisely at the points $z = m\pi + n\pi\tau$, where m and n are integers—this is an immediate consequence of

Jacobi's triple product identity [11, pp. 469, 472]. Thus, $G(z_1)$ has simple zeros at $z_1 = \pm z_2 + m\pi + n\pi\tau$, and these are the only zeros of G.

It follows that the quotient $F(z_1)/G(z_1)$ is an entire doubly periodic function with periods π and $\pi\tau$. By Liouville's theorem it is constant. If we set $z_1 = z_3$, we find that the value of the constant is zero. Thus $F(z_1) \equiv 0$ and this completes the first proof of (1).

Remark 2.1. Recently, the second author [10] used similar techniques to construct several infinite families of determinant identities involving theta functions, which are equivalent to the Macdonald identities.

3. Second proof of the Gosper–Schroeppel identity

For the second proof, we rely on two simple lemmas. As noted at the beginning of the previous section, it suffices to prove Theorem 1.1 for the case $r = s = 1$.

Lemma 3.1.

$$\theta_1(w - z|\tau)\theta_1(w + z|\tau) = \theta_3(2w|2\tau)\theta_2(2z|2\tau) - \theta_2(2w|2\tau)\theta_3(2z|2\tau).$$

Proof. By the definition of θ_1, we have

$$\theta_1(w - z|\tau)\theta_1(w + z|\tau) = -\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{m+n}q^{m^2+n^2+m+n+\frac{1}{2}}e^{2(m+n+1)iw+2(n-m)iz}.$$

Now apply the series rearrangement

$$\sum_m\sum_n c_{m,n} = \sum_j\sum_k c_{j+k,k-j} + \sum_j\sum_k c_{j+k+1,k-j} \tag{6}$$

to complete the proof. □

Lemma 3.2. *Let r_i, s_i, t_i and u_i be complex variables, where $1 \leqslant i \leqslant 3$. Then*

$$\det(r_j s_k + t_j u_k)_{1\leqslant j,k\leqslant 3} = 0.$$

Proof. Expand the determinant and observe that all terms cancel. □

We are now ready for the second proof of Theorem 1.1.

Second proof of Theorem 1.1. By Lemmas 3.1 and 3.2, we have

$$\det\big(\theta_1(w_j - z_k|\tau)\theta_1(w_j + z_k|\tau)\big)_{1\leqslant j,k\leqslant 3} = \det\big(\theta_3(2w_j|2\tau)\theta_2(2z_k|2\tau) - \theta_2(2w_j|2\tau)\theta_3(2z_k|2\tau)\big)_{1\leqslant j,k\leqslant 3} = 0. \qquad \square$$

Remark 3.3. Lemma 3.1 may also be used to give an alternative proof of Corollary 1.2. A sketch of the argument is as follows. When $s = 1$, apply Lemma 3.1 to each matrix entry, expand the determinant, and factor the resulting expression. When $s = 2, 3$ or 4, use the translational properties, as explained at the beginning of Section 2.

4. Cubic theta functions

Let q be a complex number which satisfies $|q| < 1$, and let x and y be nonzero complex numbers. The cubic theta function $a(x, y; q)$ is defined by

$$a(x, y; q) = \sum_m\sum_n q^{m^2+mn+n^2}x^{m-n}y^{m+n}$$

where the sums are over all integer values of m and n. The Hirschhorn–Garvan–Borwein cubic theta functions may be defined by[1]

$$a(x; q) = a(x, 1; q), \qquad b(x; q) = a(\omega, x; q), \quad \text{where } \omega = \exp(2\pi i/3),$$
$$c(x; q) = q^{1/3}a(x, q; q), \qquad d(x; q) = a(1, x; q).$$

[1] The functions $a(x; q)$ and $b(x; q)$ defined here correspond to the functions $a(q, x)$ and $b(q, x)$, respectively, in [9]. The function $c(x; q)$ defined here differs from the function $c(q, x)$ in [9] by a factor of $q^{1/3}$, and our function $d(x; q)$ is the same as the function $a'(q, x)$ in [9].

Explicitly, we have

$$a(x;q)=\sum_m\sum_n q^{m^2+mn+n^2}x^{m-n},\qquad b(x;q)=\sum_m\sum_n q^{m^2+mn+n^2}\omega^{m-n}x^n,$$
$$c(x;q)=\sum_m\sum_n q^{(m+\frac13)^2+(m+\frac13)(n+\frac13)+(n+\frac13)^2}x^{m-n},\qquad d(x;q)=\sum_m\sum_n q^{m^2+mn+n^2}x^n.$$

The results for $a(x;q)$ and $c(x;q)$ follow immediately from the definitions. For the function $b(x;q)$, put $m=-j$ and $n=j+k$ to get

$$b(x;q)=\sum_m\sum_n q^{m^2+mn+n^2}\omega^{m-n}x^{m+n}=\sum_j\sum_k q^{j^2+jk+k^2}\omega^{-2j-k}x^k=\sum_j\sum_k q^{j^2+jk+k^2}\omega^{j-k}x^k,$$

and the result for $d(x;q)$ is obtained similarly. When $x=1$, write

$$a(q)=a(1;q)=d(1;q),\qquad b(q)=b(1;q)\quad\text{and}\quad c(q)=c(1;q).$$

The cubic analogue of Theorem 1.1 that we shall prove is

Theorem 4.1. *Let x_1, x_2, x_3, y_1, y_2 and y_3 be complex variables. Then*

$$\det\begin{pmatrix} a(x_1,y_1;q) & a(x_1,y_2;q) & a(x_1,y_3;q)\\ a(x_2,y_1;q) & a(x_2,y_2;q) & a(x_2,y_3;q)\\ a(x_3,y_1;q) & a(x_3,y_2;q) & a(x_3,y_3;q)\end{pmatrix}=0.$$

The proof of Theorem 4.1 relies on the following lemma.

Lemma 4.2. *Let w and z be complex numbers and put $x=e^{iw}$ and $y=e^{iz}$. Recall that $q=e^{i\pi\tau}$. Then*

$$a(x,y;q)=\theta_2(w|\tau)\theta_2(z|3\tau)+\theta_3(w|\tau)\theta_3(z|3\tau).$$

Proof. This follows immediately from the series rearrangement (6) with $c_{m,n}=q^{m^2+mn+n^2}x^{m-n}y^{m+n}$. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. For $1\leqslant j\leqslant 3$, let w_j and z_j be any complex numbers for which $x_j=e^{iw_j}$ and $y_j=e^{iz_j}$. Applying Lemma 4.2 and then Lemma 3.2, we get

$$\det\bigl(a(x_j,y_k;q)\bigr)_{1\leqslant j,k\leqslant 3}=\det\bigl(\theta_2(w_j|\tau)\theta_2(z_k|3\tau)+\theta_3(w_j|\tau)\theta_3(z_k|3\tau)\bigr)_{1\leqslant j,k\leqslant 3}=0. \qquad \square$$

Remark 4.3. The proof of Theorem 4.1 that we have given is analogous to the proof in Section 3. It is also possible to give a proof which is similar to the one in Section 2. The relevant functional equations are

$$a(x,y;q)=qx^2a(qx,y;q)=qy^2a\bigl(x,q^3y;q\bigr).$$

These can be proved by replacing (m,n) with $(m+1,n-1)$ or $(m+1,n+1)$, respectively, in the definition of $a(x,y;q)$.

The next goal is to give a 2×2 analogue of Theorem 4.1.

Theorem 4.4. *Let w_j and z_j be complex variables, where $j=1$ or 2. Let $x_j=e^{iw_j}$ and $y_j=e^{iz_j}$, and recall that $q=e^{i\pi\tau}$. Then*

$$\det\begin{pmatrix} a(x_1,y_1;q) & a(x_1,y_2;q)\\ a(x_2,y_1;q) & a(x_2,y_2;q)\end{pmatrix}=\theta_1\left(\frac{w_1-w_2}{2}\,\middle|\,\frac{\tau}{2}\right)\theta_1\left(\frac{w_1+w_2}{2}\,\middle|\,\frac{\tau}{2}\right)\theta_1\left(\frac{z_1-z_2}{2}\,\middle|\,\frac{3\tau}{2}\right)\theta_1\left(\frac{z_1+z_2}{2}\,\middle|\,\frac{3\tau}{2}\right).$$

Proof. If we apply Lemma 4.2, expand the resulting determinant and simplify, we obtain

$$\begin{aligned}\det\bigl(a(x_j,y_k;q)\bigr)_{1\leqslant j,k\leqslant 2}&=\det\bigl(\theta_2(w_j|\tau)\theta_2(z_k|3\tau)+\theta_3(w_j|\tau)\theta_3(z_k|3\tau)\bigr)_{1\leqslant j,k\leqslant 2}\\ &=\theta_3(w_1|\tau)\theta_3(z_1|3\tau)\theta_2(w_2|\tau)\theta_2(z_2|3\tau)+\theta_3(w_2|\tau)\theta_3(z_2|3\tau)\theta_2(w_1|\tau)\theta_2(z_1|3\tau)\\ &\quad-\theta_3(w_1|\tau)\theta_3(z_2|3\tau)\theta_2(w_2|\tau)\theta_2(z_1|3\tau)-\theta_3(w_2|\tau)\theta_3(z_1|3\tau)\theta_2(w_1|\tau)\theta_2(z_2|3\tau).\end{aligned}$$

If we factorize the last expression and then apply Lemma 3.1, we get

$$\det\begin{pmatrix} a(x_1, y_1; q) & a(x_1, y_2; q) \\ a(x_2, y_1; q) & a(x_2, y_2; q) \end{pmatrix}$$
$$= \big(\theta_3(w_1|\tau)\theta_2(w_2|\tau) - \theta_3(w_2|\tau)\theta_2(w_1|\tau)\big)\big(\theta_3(z_1|3\tau)\theta_2(z_2|3\tau) - \theta_3(z_2|3\tau)\theta_2(z_1|3\tau)\big)$$
$$= \theta_1\left(\frac{w_1 - w_2}{2}\middle|\frac{\tau}{2}\right)\theta_1\left(\frac{w_1 + w_2}{2}\middle|\frac{\tau}{2}\right)\theta_1\left(\frac{z_1 - z_2}{2}\middle|\frac{3\tau}{2}\right)\theta_1\left(\frac{z_1 + z_2}{2}\middle|\frac{3\tau}{2}\right). \qquad \square$$

The following identities are ready consequences of Theorems 4.1 and 4.4.

Corollary 4.5. *Let w_1, w_2, w_3 and z be complex variables and recall that $q = e^{i\pi\tau}$. Then*

$$a(e^{iw_1}, e^{iz}; q)\,\theta_1\left(\frac{w_2 - w_3}{2}\middle|\frac{\tau}{2}\right)\theta_1\left(\frac{w_2 + w_3}{2}\middle|\frac{\tau}{2}\right) + a(e^{iw_2}, e^{iz}; q)\,\theta_1\left(\frac{w_3 - w_1}{2}\middle|\frac{\tau}{2}\right)\theta_1\left(\frac{w_3 + w_1}{2}\middle|\frac{\tau}{2}\right)$$
$$+ a(e^{iw_3}, e^{iz}; q)\,\theta_1\left(\frac{w_1 - w_2}{2}\middle|\frac{\tau}{2}\right)\theta_1\left(\frac{w_1 + w_2}{2}\middle|\frac{\tau}{2}\right) = 0$$

and

$$a(e^{iz}, e^{iw_1}; q)\,\theta_1\left(\frac{w_2 - w_3}{2}\middle|\frac{3\tau}{2}\right)\theta_1\left(\frac{w_2 + w_3}{2}\middle|\frac{3\tau}{2}\right) + a(e^{iz}, e^{iw_2}; q)\,\theta_1\left(\frac{w_3 - w_1}{2}\middle|\frac{3\tau}{2}\right)\theta_1\left(\frac{w_3 + w_1}{2}\middle|\frac{3\tau}{2}\right)$$
$$+ a(e^{iz}, e^{iw_3}; q)\,\theta_1\left(\frac{w_1 - w_2}{2}\middle|\frac{3\tau}{2}\right)\theta_1\left(\frac{w_1 + w_2}{2}\middle|\frac{3\tau}{2}\right) = 0.$$

Proof. Expand the 3×3 determinant in Theorem 4.1 along the first column to get

$$a(e^{iw_1}, e^{iz_1}; q)\det\begin{pmatrix} a(e^{iw_2}, e^{iz_2}; q) & a(e^{iw_2}, e^{iz_3}; q) \\ a(e^{iw_3}, e^{iz_2}; q) & a(e^{iw_3}, e^{iz_3}; q) \end{pmatrix} - a(e^{iw_2}, e^{iz_1}; q)\det\begin{pmatrix} a(e^{iw_1}, e^{iz_2}; q) & a(e^{iw_1}, e^{iz_3}; q) \\ a(e^{iw_3}, e^{iz_2}; q) & a(e^{iw_3}, e^{iz_3}; q) \end{pmatrix}$$
$$+ a(e^{iw_3}, e^{iz_1}; q)\det\begin{pmatrix} a(e^{iw_1}, e^{iz_2}; q) & a(e^{iw_1}, e^{iz_3}; q) \\ a(e^{iw_2}, e^{iz_2}; q) & a(e^{iw_2}, e^{iz_3}; q) \end{pmatrix} = 0.$$

Now apply Theorem 4.4 to each of the 2×2 determinants, and cancel the common factor of $\theta_1(\frac{z_2-z_3}{2}|\frac{3\tau}{2})\theta_1(\frac{z_2+z_3}{2}|\frac{3\tau}{2})$ that arises. This proves the first result. The second result can be obtained in a similar way, by expanding along the first row. $\square$

5. Applications

In this section we consider some special cases of results in the previous section. In Theorem 5.2 we obtain an identity that is equivalent to a result of Hirschhorn, Garvan and Borwein [9, (1.21)]. By specializing further, and making use of infinite product formulas for $b(q)$ and $c(q)$, we obtain a fundamental result from the theory of elliptic functions in signature 3 in Theorem 5.5.

Theorem 5.1. *Let*

$$F(x, y; q) = \frac{(x - y)(1 - xy)}{xy}\prod_{j=1}^{\infty}(1 - q^j xy)(1 - q^j x^{-1}y^{-1})(1 - q^j xy^{-1})(1 - q^j x^{-1}y).$$

The Hirschhorn–Garvan–Borwein functions satisfy the identities

$$\det\begin{pmatrix} a(x; q) & c(x; q) \\ a(y; q) & c(y; q) \end{pmatrix} = q^{1/3}F(x, y; q)\prod_{j=1}^{\infty}(1 - q^j)^4, \qquad \det\begin{pmatrix} b(x; q) & d(x; q) \\ b(y; q) & d(y; q) \end{pmatrix} = 3qF(x, y; q^3)\prod_{j=1}^{\infty}(1 - q^{3j})^4.$$

Proof. The infinite product for θ_1 [11, p. 470] may be given as

$$\theta_1\left(\frac{z}{2}\middle|\frac{\tau}{2}\right) = -iq^{1/8}(x^{1/2} - x^{-1/2})\prod_{j=1}^{\infty}(1 - q^j x)(1 - q^j x^{-1})(1 - q^j),$$

where $x = e^{iz}$. Using this and the definition of F we find that Theorem 4.4 may be written in the form

$$\det\begin{pmatrix} a(x_1,y_1;q) & a(x_1,y_2;q) \\ a(x_2,y_1;q) & a(x_2,y_2;q)\end{pmatrix} = qF(x_1,x_2;q)F\big(y_1,y_2;q^3\big)\prod_{j=1}^{\infty}\big(1-q^j\big)^2\big(1-q^{3j}\big)^2.$$

The results in Theorem 5.1 follow from this by taking $(x_1,x_2,y_1,y_2)=(x,y,1,q)$ and $(x_1,x_2,y_1,y_2)=(\omega,1,x,y)$, respectively, and using the results

$$qF\big(1,q;q^3\big)=\prod_{j=1}^{\infty}\frac{(1-q^j)^2}{(1-q^{3j})^2} \quad\text{and}\quad F(\omega,1;q)=3\prod_{j=1}^{\infty}\frac{(1-q^{3j})^2}{(1-q^j)^2}. \qquad \square$$

The first result in the next theorem is equivalent to formula (1.21) in [9,10], and the second result is (6.4) in [4].

Theorem 5.2.

$$\det\begin{pmatrix} a(x;q) & c(x;q) \\ a(q) & c(q)\end{pmatrix} = q^{1/3}\big(2-x-x^{-1}\big)\prod_{j=1}^{\infty}\big(1-q^jx\big)^2\big(1-q^jx^{-1}\big)^2\big(1-q^j\big)^4$$

and

$$\det\begin{pmatrix} b(x;q) & d(x;q) \\ b(q) & a(q)\end{pmatrix} = 3q\big(2-x-x^{-1}\big)\prod_{j=1}^{\infty}\big(1-q^{3j}x\big)^2\big(1-q^{3j}x^{-1}\big)^2\big(1-q^{3j}\big)^4.$$

Proof. Take $y=1$ in Theorem 5.1. $\square$

The next goal is to let $x\to 1$ in the results in Theorem 5.2. We will need

Lemma 5.3.

$$\frac{\partial^2}{\partial x^2}a(x;q)\bigg|_{x=1} = 2q\frac{d}{dq}a(q), \qquad \frac{\partial^2}{\partial x^2}b(x;q)\bigg|_{x=1} = \frac{2}{3}q\frac{d}{dq}b(q),$$
$$\frac{\partial^2}{\partial x^2}c(x;q)\bigg|_{x=1} = 2q\frac{d}{dq}c(q) \quad \textit{and} \quad \frac{\partial^2}{\partial x^2}d(x;q)\bigg|_{x=1} = \frac{2}{3}q\frac{d}{dq}a(q).$$

Proof. Clearly

$$\sum_m\sum_n m^2q^{m^2+mn+n^2} = \sum_m\sum_n n^2q^{m^2+mn+n^2}. \tag{7}$$

If we replace (m,n) with $(m,-m-n)$ we get

$$\sum_m\sum_n n^2q^{m^2+mn+n^2} = \sum_m\sum_n (m+n)^2q^{m^2+mn+n^2}. \tag{8}$$

From (7) and (8) it follows that

$$\sum_m\sum_n m^2q^{m^2+mn+n^2} = -2\sum_m\sum_n mnq^{m^2+mn+n^2}. \tag{9}$$

Using (7) and (9) we have

$$\begin{aligned}\frac{\partial^2}{\partial x^2}a(x;q)\bigg|_{x=1} &= \sum_m\sum_n (m-n)^2q^{m^2+mn+n^2} = 3\sum_m\sum_n m^2q^{m^2+mn+n^2} \\ &= 2\sum_m\sum_n \big(m^2+mn+n^2\big)q^{m^2+mn+n^2} = 2q\frac{d}{dq}a(q).\end{aligned}$$

This proves the first result. The other results may be proved using the same procedure. The only significant difference is that for the result involving $c(q)$, we replace (m,n) with $(m,-m-n-1)$ to obtain the analogue of (8). We omit the details. $\square$

Theorem 5.4. *Let D be the differential operator defined by $Df=q\frac{df}{dq}$. Then*

$$\det\begin{pmatrix} a(q) & c(q) \\ Da(q) & Dc(q)\end{pmatrix} = q^{1/3}\prod_{j=1}^{\infty}\big(1-q^j\big)^8 \quad \textit{and} \quad \det\begin{pmatrix} a(q) & b(q) \\ Da(q) & Db(q)\end{pmatrix} = -9q\prod_{j=1}^{\infty}\big(1-q^{3j}\big)^8.$$

Proof. Divide the first result in Theorem 5.2 by $(1-x)^2$ and let $x \to 1$ to obtain

$$\lim_{x\to 1} \frac{-x}{(1-x)^2} \det\begin{pmatrix} a(x;q) & c(x;q) \\ a(q) & c(q) \end{pmatrix} = q^{1/3} \prod_{j=1}^{\infty} \left(1-q^j\right)^8.$$

Interchange the rows and apply L'Hôpital's rule twice to get

$$\frac{1}{2} \det\left.\begin{pmatrix} a(q) & c(q) \\ \frac{\partial^2}{\partial x^2} a(x;q) & \frac{\partial^2}{\partial x^2} c(x;q) \end{pmatrix}\right|_{x=1} = q^{1/3} \prod_{j=1}^{\infty} \left(1-q^j\right)^8.$$

Now apply Lemma 5.3 to obtain the first result in Theorem 5.4. The proof of the second result in Theorem 5.4 is similar. We omit the details, except to say that the negative sign arises from interchanging the two columns in the matrix (as well as interchanging the two rows). □

Theorem 5.4 leads to a simple proof of the following fundamental formula from Ramanujan's theory of elliptic functions in signature 3.

Theorem 5.5. *Let $z = a(q)$ and $x = \frac{c^3(q)}{a^3(q)}$. Then*

$$q\frac{dx}{dq} = z^2 x(1-x).$$

Proof. The proof uses the Borweins' cubic identity

$$a^3(q) = b^3(q) + c^3(q) \tag{10}$$

as well as the infinite products

$$b(q) = \prod_{j=1}^{\infty} \frac{(1-q^j)^3}{(1-q^{3j})} \quad \text{and} \quad c(q) = 3q^{1/3} \prod_{j=1}^{\infty} \frac{(1-q^{3j})^3}{(1-q^j)}. \tag{11}$$

Many proofs of (10) and (11) have been published. For example, the proofs in [7] are simple and self-contained.

Let us write a, b and c for $a(q)$, $b(q)$ and $c(q)$, respectively. By (11), the first identity in Theorem 5.4 is equivalent to

$$aq\frac{dc}{dq} - cq\frac{da}{dq} = \frac{b^3 c}{3}.$$

Multiply by $3c^2/a^4$ and use (10) to get

$$q\frac{d}{dq}\left(\frac{c^3}{a^3}\right) = a^2\left(\frac{c^3}{a^3}\right)\left(\frac{b^3}{a^3}\right) = a^2\left(\frac{c^3}{a^3}\right)\left(1-\frac{c^3}{a^3}\right).$$

This is equivalent to the identity in the statement of the theorem. □

Other proofs of Theorem 5.5, by a variety of methods, have been given in [1, (4.4)], [2, (4.4)], [3, (4.7)], [5, (11.13)] and [6, Theorem 4.1]. The proof, we have given above, is to observe that the identity is essentially a special case of a matrix determinant.

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